Linear Algebra 1

09/10/2026

You are **NOT** allowed to use any type of calculators.

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1 Linear equations $(4+3+3+5+2+4+2+2=25$ pts)

A drone is flying in three-dimensional space. Its unknown position, relative to a chosen reference point, is

$$P = (x, y, z).$$

Recall that if $P = (x, y, z)$ and $Q = (a, b, c)$, then the distance between P and Q is

$$\operatorname{dist}(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Consequently, the *squared distance* between the two points is

$$\operatorname{dist}(P, Q)^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

Four positioning beacons are located at

$$Q_0 = (0, 0, 0), \qquad Q_1 = (1, 2, 1), \qquad Q_2 = (-2, 1, 3), \qquad Q_3 = (3, -1, -2).$$

The measured squared distances between the drone and the beacons are

$$\operatorname{dist}(P, Q_0)^2 = 14, \qquad \operatorname{dist}(P, Q_1)^2 = 14, \qquad \operatorname{dist}(P, Q_2)^2 = 20, \qquad \operatorname{dist}(P, Q_3)^2 = 26.$$

(a) Write the four equations determined by the distance measurements.

(b) The resulting equations are not initially linear. Subtract the equation corresponding to Q_0 from each of the other three equations.

(c) Show that all quadratic terms cancel, and write the resulting linear system in the form

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boldsymbol{b}.$$

(d) Use elementary row operations to put the augmented matrix into row echelon form.

(e) Determine whether the system is consistent. Justify your answer.

(f) Continue using elementary row operations to obtain the reduced row echelon form of the augmented matrix.

(g) Identify the lead and free variables.

(h) Obtain the general solution.

2 Determinants and nonsingularity (16 + 4 = 20 pts)

Let

$$M = \begin{bmatrix} a+1 & a & a & a \\ b & b+1 & b & b \\ c & c & c+1 & c \\ d & d & d & d+1 \end{bmatrix}.$$

(a) Calculate $\det(M)$ using elementary row and column operations. Clearly indicate how each operation affects the determinant.

(b) Determine the values of a, b, c, d for which M is invertible.

3 Eigenvalues and eigenvectors (7 + 6 + 2 + 5 = 20 pts)

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$. Thus, AB is an $m \times m$ matrix whereas BA is an $n \times n$ matrix.

(a) Suppose that $\lambda \neq 0$ is an eigenvalue of AB with corresponding eigenvector v. Prove that Bv is an eigenvector of BA corresponding to λ.

(b) Prove the converse: if $\lambda \neq 0$ is an eigenvalue of BA, then λ is also an eigenvalue of AB.

(c) Conclude that AB and BA have the same nonzero eigenvalues.

(d) Explain why the condition $\lambda \neq 0$ is necessary.

4 Diagonalization (5 + 8 + 7 + 5 = 25 pts)

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(a) Determine the characteristic polynomial of A.

(b) Decide whether A is diagonalizable. Justify your answer.

(c) If A is diagonalizable, find a nonsingular matrix X and a diagonal matrix D such that

$$A = XDX^{-1}.$$

(d) Use the diagonalization to calculate A^n for every positive integer n.

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10 pts free